Entry Task: Evaluate

$$\int \sqrt{9+x^2}\,dx$$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$$\int \sqrt{9+9\tan^2\theta}\;3\sec^2\theta\,d\theta$$

$9(1+\tan^2\theta)$

$9\sec^2\theta$

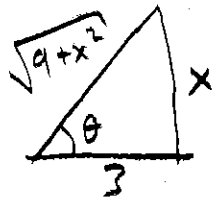
$$\int 3\sec\theta\;3\sec^2\theta\,d\theta$$

$$9\int \sec^3\theta\,d\theta$$ ↷ FROM TABLE

$$= \frac{9}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$$

$$= \frac{9}{2}\left(\frac{\sqrt{9+x^2}}{3}\cdot\frac{x}{3} + \ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|\right)+C$$

$$= \boxed{\frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln\left|\frac{\sqrt{9+x^2}+x}{3}\right| + C}$$



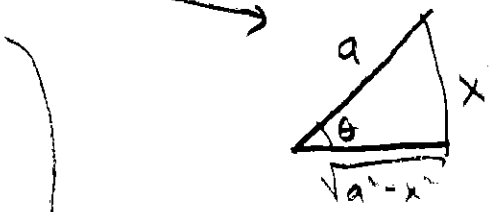
$\tan\theta = x/3$

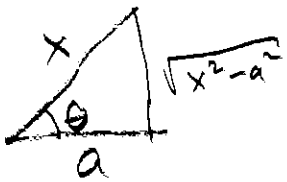
7.3 Trig. Substitution

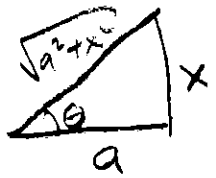
Steps-by-step:

1. Substitute as directed below. Simplify (should eliminate root) Don't forget dx.
2. Use 7.2 methods for trig integrals.
3. Draw a triangle and return to x.

CASE	SUBSTITUTION
$a^2 - x^2$	$x = a\sin(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$





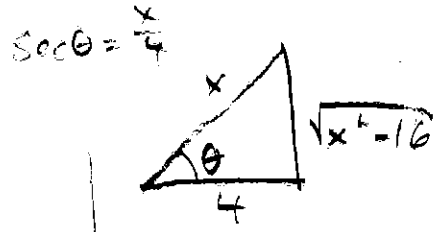


Example:

$$\int \frac{\sqrt{x^2-16}}{x}\,dx$$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$



$$\int \frac{\sqrt{16\sec^2\theta - 16}}{4\sec\theta}\,4\sec\theta\tan\theta\,d\theta$$

$\sqrt{16(\sec^2\theta - 1)}$

$\sqrt{16\tan^2\theta}$

$4\tan\theta$

$\theta = \sec^{-1}(x/4)$

$$\int 4\tan\theta\cdot\tan\theta\,d\theta$$

$$4\int \tan^2\theta\,d\theta$$

$$4\int \sec^2\theta - 1\,d\theta$$

$$4(\tan\theta - \theta) + C$$

$$4\tan\theta - 4\theta + C$$

NOTE: IT IS NOT ACCEPTABLE TO WRITE $\tan(\sec^{-1}(x/4))$

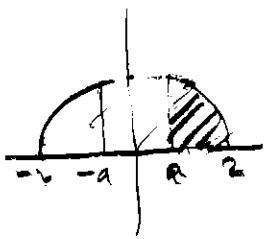
↑ NEVER WRITE THIS.

$$4\frac{\sqrt{x^2-16}}{4} - 4\sec^{-1}(x/4) + C = \boxed{\sqrt{x^2-16} - 4\sec^{-1}(x/4) + C}$$

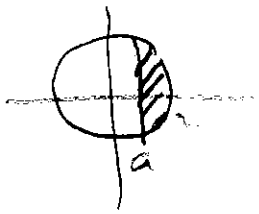
mportant Application

Area under a circle

$$\int \sqrt{4-x^2}\,dx$$

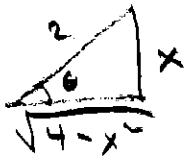


$x^2+y^2=4$



$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$



$\sin\theta = x/2$

$$= \int \sqrt{4-4\sin^2\theta}\; 2\cos\theta\,d\theta$$

$$= \int 2\cos\theta\; 2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta$$

$$= 4\int \tfrac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= 2\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C$$

$$= 2\theta + \sin(2\theta) + C$$

??? ↓ HALF-ANGLE

$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\tfrac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$= \boxed{2\sin^{-1}\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\,x\sqrt{4-x^2} + C}$$

$\sqrt{4(1-\sin^2\theta)}$

$\sqrt{4\cos^2\theta}$

$2\cos\theta$

CUT A CIRCLE IN THIRDS?

$$2\int_a^2 \sqrt{4-x^2}\,dx \overset{?}{=} \tfrac{1}{3}\pi(2)^2$$

$$2\sin^{-1}\left(\tfrac{x}{2}\right) + \tfrac{1}{2}x\sqrt{4-x^2}\,\Big|_a^2 \overset{?}{=} \tfrac{2}{3}\pi$$

$$\left(2\cdot\tfrac{\pi}{2} + 0\right) - \left(2\sin^{-1}\left(\tfrac{a}{2}\right) + \tfrac{1}{2}a\sqrt{4-a^2}\right) \overset{?}{=} \tfrac{2}{3}\pi$$

Solver $\Rightarrow a \approx 0.529864$

Perfect Squares

$(x+3)^2 = x^2+6x+9$ (SQUARED; HALF)

$(x-5)^2+2 = x^2-10x+25+2$

Completing the Square

$x^2+16x+64-64 = (x+8)^2-64$ (HALF; SQUARE)

$3x^2-12x = 3(x^2-4x+4-4) = 3((x-2)^2-4)$ (HALF -2)

Given $\sqrt{ax^2+bx+c}$

Complete square!

- *Factor out "a"*
- *Add/subtract half-middle squared*

Example:

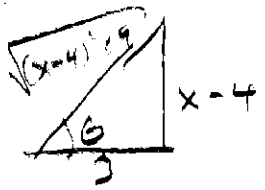
$$\int \sqrt{4x^2-32x+100}\,dx$$

$$= \int \sqrt{4(x^2-8x+25)}\,dx$$

$$= \frac{1}{2}\int \sqrt{x^2-8x+25}\,dx$$

$x^2-8x+16-16+25 = (x-4)^2+9$ (HALF $=-4$; SQUARE)

$$= 2\int \sqrt{(x-4)^2+9}\,dx$$



$x-4 = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{9\tan^2\theta+9}$

$\sqrt{9(\tan^2\theta+1)}$

$\sqrt{9\sec^2\theta}$

$$= 2\int \frac{1}{3\sec\theta}\,3\sec^2\theta\,d\theta$$

$$= 2\int \sec\theta\,d\theta$$

$$= 2\ln|\sec\theta+\tan\theta| + C$$

$$= 2\ln\left|\frac{\sqrt{(x-4)^2+9}}{3}+\frac{x-4}{3}\right| + C$$

Example:

$$\int \frac{x}{\sqrt{16-6x-x^2}}\,dx$$

$16-6x-x^2$

$16+9-9-6x-x^2$

-3 HALF

$25-(9+6x+x^2)$

$25-(x+3)^2$

$$\int \frac{x}{\sqrt{25-(x+3)^2}}\,dx$$

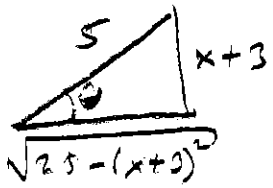
$x+3=5\sin\theta \rightarrow x=5\sin\theta-3$

$dx=5\cos\theta\,d\theta$

$$\int \frac{5\sin\theta-3}{5\cos\theta}\,5\cos\theta\,d\theta$$

$$\frac{\sqrt{25-25\sin^2\theta}}{\sqrt{25\cos^2\theta}}$$

$$= -5\cos\theta-3\theta+C$$



$$= -5\frac{\sqrt{25-(x+3)^2}}{5}-3\sin^{-1}\left(\frac{x+3}{5}\right)+C$$

$\theta=\sin^{-1}\left(\frac{x+3}{5}\right)$